are possessed by the Fourier approximations as well. As we shall see later, most of the other functions considered as potential candidates for approximating functions (sines, cosines, exponentials, etc.) must themselves be evaluated by using approximations; almost invariably, these approximations are given in terms of polynomials or ratios of polynomials.

All these obvious advantages of the polynomials would be of little value if there were no analytical justification for believing that polynomials can, in fact, yield good approximations for a given function $f(x)$. Here, "good" implies that the discrepancy between an approximating polynomial $p_n(x)$ and $f(x)$, that is, the error in the approximation, can be made arbitrarily small. Fortunately, this theoretical justification does exist. *Any* continuous function $f(x)$ can be approximated to *any* desired degree of accuracy on a specified closed interval by *some* polynomial $p_n(x)$. This follows from the *Weierstrass approximation theorem* stated here without proof [6]:

> If $f(x)$ is continuous in the closed interval $[a,b]$, (that is, $a \leqslant x \leqslant b$) then, given any $\epsilon > 0$, there is some polynomial $p_n(x)$ of degree $n \equiv n(\epsilon)$ such that
>
> $$|f(x) - p_n(x)| < \epsilon, \qquad a \leqslant x \leqslant b.$$

Unfortunately, although it is reassuring to know that *some* polynomial will approximate $f(x)$ to a specified accuracy, the usual criteria for generating approximating polynomials in no way guarantee that the polynomial found is the one which the Weierstrass theorem shows must exist. If $f(x)$ is in fact unknown except for a few sampled values, then the theorem is of little relevance. (It is comforting nonetheless!)

The case for polynomials as approximating functions is not so strong that other possibilities should be ruled out completely. Periodic functions can often be approximated very efficiently with Fourier functions; functions with an obvious exponential character will be described more compactly with a sum of exponentials, etc. Nevertheless, for the general approximation problem, polynomial approximations are usually adequate and reasonably easy to generate.

The remainder of this chapter will be devoted to polynomial approximations of the form

$$f(x) \doteq p_n(x) = \sum_{i=0}^{n} a_i x^i. \qquad (1.1)$$

For a thorough discussion of several other approximating functions, see Hamming [2].

1.3 Polynomial Approximation—A Survey

After selection of an nth-degree polynomial (1.1) as the approximating function, we must choose the criterion for "fitting the data." This is equivalent to establishing the procedure for computing the values of the coefficients $a_0, a_1, \ldots, a_n$.

The Interpolating Polynomial. Given the paired values $(x_i, f(x_i))$, $i = 0, 1, \ldots, n$, perhaps the most obvious criterion for determining the coefficients of $p_n(x)$ is to require that

$$p_n(x_i) = f(x_i), \qquad i = 0, 1, \ldots, n. \qquad (1.2)$$

Thus the nth degree polynomial $p_n(x)$ must reproduce $f(x)$ *exactly* for the $n+1$ arguments $x = x_i$. This criterion seems especially pertinent since (from a fundamental theorem of algebra) there is one and only one polynomial of degree n or less which assumes specified values for $n+1$ distinct arguments. This polynomial, called the ***nth degree interpolating polynomial***, is illustrated schematically for $n = 3$ in Fig. 1.1. Note that requirement (1.2)

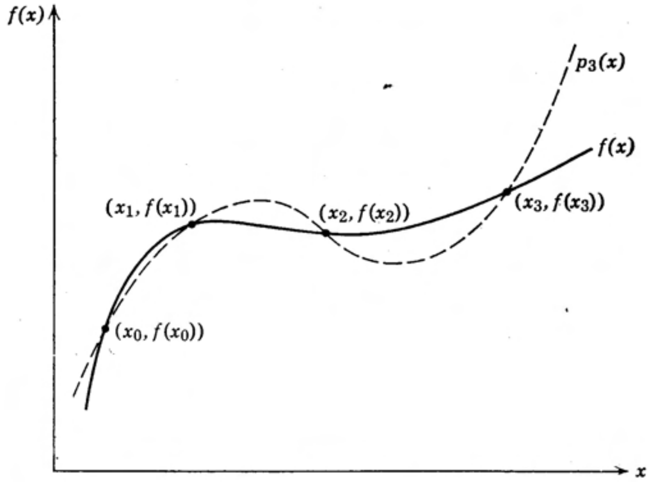

***Figure 1.1** The interpolating polynomial.*

establishes the value of $p_n(x)$ for *all* x, but in no way guarantees accurate approximation of $f(x)$ for $x \neq x_i$, that is, for arguments other than the given base points. If $f(x)$ should be a polynomial of degree n or less, agreement is of course *exact* for all x.

The interpolating polynomial will be developed in considerable detail in Sections 1.5 to 1.9.

The Least-Squares Polynomial. If there is some question as to the accuracy of the individual values $f(x_i)$, $i = 0, 1, \ldots, n$ (often the case with experimental data), then it may be unreasonable to require that a polynomial fit the $f(x_i)$ exactly. In addition, it often happens that the desired polynomial is of low degree, say m, but that there are many data values available, so that $n > m$. Since the exact matching criterion of (1.2) for $n+1$ functional values can be satisfied only by one polynomial of degree n or less, it is generally impossible to find an interpolating polynomial of degree m using all $n+1$ of the sampled functional values.

Some other measure of goodness-of-fit is needed. Instead of requiring that the approximating polynomial reproduce the given functional values exactly, we ask only that it fit the data as closely as possible. Of the many